FINITE DIFFERENCE METHOD ON VORONOI DIAGRAM

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ABSTRACT: The Voronoi cells and the notion of natural neighbors are used to develop the finite difference method. The Laplace (non-Sibsonian) interpolant is based on the Voronoi diagram (VD) which partitions space into closet-point regions.

In this work, Voronoi diagram is constructed first on the set of nodes and set of points. Then finite difference operator (method) is constructed on Voronoi diagram which is called Voronoi Cell Finite Difference Method (VCFDM). We provide the formulation of VCFDM for diffusion equation for general case. We show that VCFDM is the same as classical finite difference method on uniform grid. We prove for 1D (one dimension), 2D and 3D.

1. INTRODUCTION

One of the most universal and effective methods, in wide use today, for approximately solving equations of mathematics is the finite difference method (FDM). In this paper, mainly we study the finite difference method on Voronoi diagram which is called Voronoi Cell Finite Difference Method (VCFDM).

The interpolation scheme used in Voronoi Cell Finite Difference Method (VCFDM) is known as natural neighbor (n-n) interpolation. Natural neighbor interpolation relies on concepts such as Voronoi diagrams and Delaunay tessellations in computational geometry, to construct the interpolant.

We assume the discrete model of a body $\Omega \subset R^2$ consists of a set of distinct nodes N, and a polygonal description of the boundary $\partial\Omega$. In this method, the trial and test functions are constructed using natural neighbor interpolants. The Sibson and the Laplace (non-Sibsonian) interpolants which are based on natural neighbors. These interpolants are based on the Voronoi tessellation of the nodal set N. The Voronoi tessellation is a unique geometric structure that partitions the domain into disjoint nodal regions on the basis of shortest Euclidean distances. The interpolants are smooth everywhere, except at the nodes where they are C^0. In this work, a brief description of Voronoi diagrams and Delaunay triangles are first given. The Voronoi diagram and its geometric dual, the Delaunay triangulation, are well-known geometric constructs in computational geometry. Then, secondly, an extensive discussion on natural neighbor interpolants that are used in this method is presented. A detailed description of the construction, properties, and computational procedure to evaluate natural neighbor shape functions are outlined.

In this paper, we focus this method on (with) the Laplace interpolant. On regular grids, the discrete Laplacian is shown to reduce to the classical finite difference scheme. Finally, a summary of some of the observations in this work are presented, with some concluding remarks.

2. NATURAL NEIGHBOR-BASED INTERPOLANTS

Our approach depends heavily on using Voronoi diagram. We begin with a definition and its features. We present here a brief description of Voronoi diagram and Delaunay triangles in the context of natural neighbor interpolation. Figure 1 shows that Voronoi diagram as a sample Voronoi diagram.

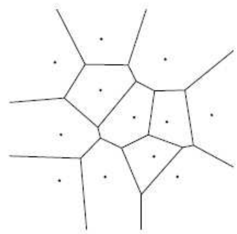

Figure 1. A sample Voronoi diagram

2.1. Voronoi Diagram (VD) and Delaunay Triangulation (DT)

The Voronoi cell for each node is formed by the intersection of perpendicular half-spaces, whereas its dual, the Delaunay tessellation, is constructed by connecting nodes that have a common (d–1) dimensional Voronoi facet.

The Voronoi diagram $V(N)$ of the set N is a subdivision of the domain into regions $V(n_I)$ such that any point in $V(n_I)$ is closer to node n_I than to any other node $n_J \in N\,(J \neq I)$. The Voronoi diagram in essence partitions space into closest-point regions. The region $V(n_I)$ (Voronoi cell) for a node n_I on the real line (R) is defined as $V(n_I) = \{x \in R : d(x, x_I) < d(x, x_J), \forall I \neq J\}$. The region $V(n_I)$ (Voronoi cell) for a node n_I within the convex hull is a convex polygon in $R^2 : V(n_I) = \{x \in R^2 : d(x, x_I) < d(x, x_J)\ \ \forall J \neq I\}$ where $d(x_I, x_J)$ is the Euclidean distance between x_I and x_J (Sukumar,2003). Also the region $V(n_I)$ (Voronoi cell) for a node n_I within the convex hull is a convex polyhedron in R^3: $V(n_I) = \{x \in R^3 : d(x, x_I) < d(x, x_J)\ \ \forall J \neq I\}$.

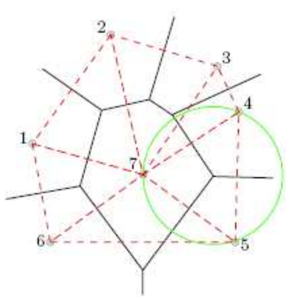

Figure 2.Voronoi Diagram containing seven nodes

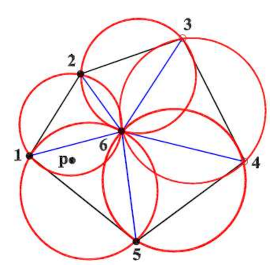

Figure 3.Delaunay Triangles DT (six nodes) and Delaunay Circumcircles

The VD is closely related to another structure which is the Delaunay triangulation (DT). The Delaunay triangulation is the dual structure of the Voronoi diagram. An important property of the Delaunay triangulation is the empty circumcircle criterion. The circumcircle is defined as the circle, that passes through all three vertices of a Delaunay triangle. This means that there is no node inside the circumcircle of any Delaunay triangle. The center of this circumcircle is the point of intersection of the three Voronoi edges, belonging to the nodes of this triangle. Two nodes in a Delaunay triangulation are called natural neighbors, if and only if there is a triangle edge, which connects both nodes.

2.2. Delaunay Tessellation

The Delaunay tessellation within the set N is a partition of the convex hull Ω of all the nodes into regions Ω_i such that $\Omega = \cup \Omega_i$, where each Ω_i is the tetrahedron defined by four nodes of the same Voronoi sphere.
The Delaunay tessellation of a set of nodes is non-unique. For the same node distribution, different tetrahedrations are possible. Therefore, a partition based on the Delaunay tessellation is sensitive to geometric perturbations of the node positions. On the other hand, its dual, the Voronoi diagram, is unique. Thus, it makes more sense to define a partition based on the unique Voronoi diagram than on Delaunay tessellations.

2.3. Natural Neighbor Interpolants

There are two basic types of interpolants. They are Sibson interpolant and Laplace interpolant.

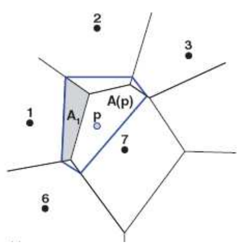

Figure 5. Sibson interpolant

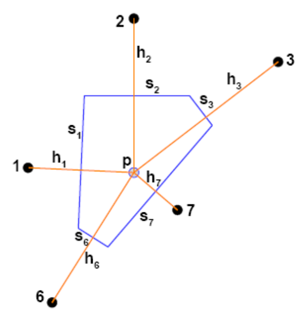

Figure 6. Laplace interpolant

The Sibson interpolant is based on Voronoi diagram and is defined by the ratio of area measures in 2D. In mathematical form, Sibson shape function for node I is defined as $\phi_I(x) = \dfrac{A_I(x)}{A(x)}$,

$A(x) = \sum_{J=1}^{n} A_J(x)$, where $A(x)$ is the area of the first-order Voronoi cell of p and $A_I(x)$ is the area of overlap between the first-order Voronoi cells of p and node I. Sibson shape functions are non-negative, interpolate nodal data, form a partition of unity, and satisfy the local co-ordinate property (linear precision), $0 \le \phi_I(x) \le 1$, $\phi_I(x) = \delta_{IJ}$, $\sum_{I=1}^{n} \phi_I(x) = 1$ in Ω, δ_{IJ} is the Cronecker delta with the conditions if $I = J$, its value is one and zero otherwise (Sukumar,1998). Natural neighbor shape functions also satisfy the local co-ordinate property, namely $x = \sum_{I=1}^{n} \phi_I(x)\, x_I$ and hence the Sibson interpolant can exactly represent any linear field, which is known as linear completeness in the finite element literature.
The Laplace interpolant is a natural neighbor-based interpolation scheme that is based on the underlying Voronoi diagram and Delaunay triangulation. For a set N consisting of n nodes with locations $\{x_i\}_{i=1}^{n}$, the natural neighbors of a point p within the convex hull of N are

defined through the Delaunay circumcircles. In 2D, the Laplace shape function for node I is defined as

$$\phi_J(x) = \frac{\alpha_J(x)}{\sum_{J=1}^{n} \alpha_J(x)}, \; \alpha_J(x) = \frac{s_J(x)}{h_J(x)}, \; x \in R^2 \tag{1}$$

where $\alpha_J(x)$ is the Laplace weight function, $s_I(x)$ is the length of the Voronoi edge associated with p and node I, and $h_I(x)$ is the Euclidean distance between p and node I.

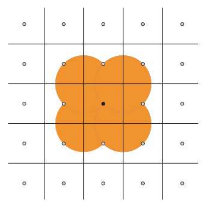

Figure 7**.** Support of natural neighbor-based shape function

In 2D, the Laplace shape function involves ratio of length measures whereas the Sibson shape function is based on the ratio of areas. The Laplace shape functions are also strictly positive, interpolate nodal data and also form a partition of unity. The domain of support (region in which $\phi_I > 0$) of the Laplace shape function ϕ_I is the union of Delaunay circumcircles about node.
The distinction of the Sibson and the Laplace shape functions is notable in two key areas: smoothness and symmetry. The Sibson shape function is continuous at everywhere whereas the Laplace shape function is continuous at nodal locations as well as on the boundary of the support. The Laplace weight function is symmetric $(\alpha_{IJ} = \alpha_{JI})$, but the Sibson weight is not (Sukumar, 2003).

3. FINITE DIFFERENCE OPERATORS ON VORONOI DIAGRAM (VCFDM)

We will study the construction of finite difference operator (method) on Voronoi diagram which is called Voronoi Cell Finite Difference Method (VCFDM). There are several types of finite difference operators. In this work, we emphasize on only two types which are Voronoi Cell Discrete Diffusion Operator (VCDDO) and Voronoi Cell Discrete Laplace Operator (VCDLO).

3.1 Voronoi Cell Discrete Diffusion Operator (VCDDO)

In this study, we define a discrete diffusion operator in 2D as follows:

$$(L_h u)_I = \frac{1}{A_I}\left[\sum_{J=1}^{n} \kappa_{IJ}\alpha_{IJ}u_J - \left(\sum_{J=1}^{n} \kappa_{IJ}\alpha_{IJ}\right)u_I\right] \tag{2}$$

with $$A_I = \frac{1}{4}\sum_{J=1}^{n} s_{IJ}h_{IJ} = \frac{1}{4}\sum_{J=1}^{n} \alpha_{IJ}h_{IJ}^2, \left|\alpha_{IJ} = \frac{s_{IJ}}{h_{IJ}}\right|, \alpha_I = \sum_{J=1}^{n} \alpha_{IJ}$$

where α_{IJ} is the Laplace weight function, L_h is the discrete diffusion operator, h denotes a measure of the nodal spacing, s_{IJ} is the length of the Voronoi edge associated with nodes I

and J (see Figure 6) and κ_{IJ} is the diffusive coefficient. Above these two equations can be derived from the steady-state diffusion equation with Dirichlet boundary conditions:

$$-Lu(x) = -\nabla.(\kappa(x)\nabla u(x)) = f(x) \quad \text{in } \Omega \subset R^2 \tag{3a}$$

$$u(x) = g(x) \text{ on } \partial\Omega \tag{3b}$$

where ∇ is the gradient operator, Ω is an open set in 2D, and $\partial\Omega$ is the boundary of Ω. The discrete form in Equation (3) is written as

$$-L_h u(x_I) = f(x_I), \qquad I = 1,2,...,N(x_I \in \Omega) \tag{4a}$$

$$u(x_I) = g(x_I), \; x_I \in \partial\Omega \tag{4b}$$

where N is the number of the nodes in the domain.

3.2 Voronoi Cell Discrete Laplace Operator (VCDLO)

We also define a Voronoi Cell Discrete Laplace Operator L by,

$$(L\phi)(X) = \frac{1}{V[X]}\sum_{y=x} A[X,Y]\frac{\phi(Y)-\phi(X)}{|Y-X|} \tag{5}$$

for $X \in S_N$, where $A[X,Y]$ is the length of the common edge of $P(X,S_N)$ and $P(Y,S_N)$, and $V(X)$ is the area of the Voronoi polygon (Borgers, 1987).

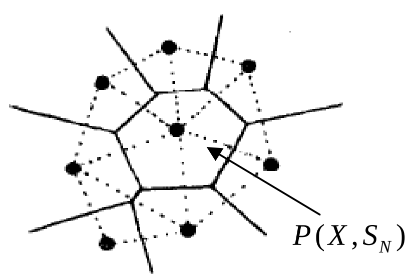

Figure 8.Voronoi diagram and Delaunay Triangulation.

The operator in Equation (5) is the stiffness matrix obtained with piecewise linear finite elements on the Delaunay triangulation. L is not pointwise consistent with the VCDLO (Borgers, 1987). In VCDDO, $(L_h u)_I$ is equivalent to $(L\phi)(X)$ in VCDLO if we take $\kappa_{IJ} = 1$.

4. VORONOI CELL FINITE DIFFERENCE OPERATOR ON USUAL GRIDS

In this section, we will consider VCFDM on usual grids, in particular VCDLO, to show that VCFDM on usual grids is the classical FDM. We will derive this in 1-D, 2-D and 3-D.

4.1 VCDLO in 1-D

We consider the finite difference approximation for the Laplacian $((Lu) = u_{xx})$ on a one-dimensional non-uniform grid.

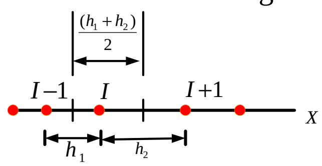

Figure 9. Voronoi cell for a non-uniform grid in one-dimension

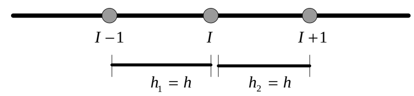

Figure 10. Uniform-Grid in 1-D

The discrete Laplacian is computed at node I, whose natural neighbors are $I-1$ and $I+1$. In this case, A_I in Equation (2) can be considered as a length of the cell l_I. The Voronoi cell of node I is shown in Figure 9, the length of the cell is $l_I = \frac{h_1 + h_2}{2}$ where $h_1 = x_I - x_{I-1}$ and $h_2 = x_{I+1} - x_I$ are the grid spacings. Using Equation (2), the discrete Laplacian $(\kappa = 1)$ for node I can be written as

$$(L_h u)_I = \frac{1}{l_I}\left[\sum_{J=1}^{2} \alpha_{IJ} u_J - \alpha_I u_I\right], \ \alpha_I = \sum_{J=1}^{2} \alpha_{IJ}. \tag{6}$$

Here, we assume $s_{IJ} = 1$ and which becomes,

$$(L_h u)_I = \frac{2}{h_1 + h_2}\left[\frac{1}{h_1} u_{I-1} + \frac{1}{h_2} u_{I+1} - \left(\frac{1}{h_1} + \frac{1}{h_2}\right) u_I\right] = \frac{2}{h_1 h_2 (h_1 + h_2)}\left[h_2 u_{I-1} + h_1 u_{I+1} - (h_1 + h_2) u_I\right]. \tag{7}$$

Equation (7) is classical-central finite difference approximation in 1-D on non-uniform grids. As a counter example, if we take $h_1 = \frac{3\Delta x}{4}$ and $h_2 = \frac{3\Delta x}{2}$, then we get $(L_h u)_I = \frac{16}{27\Delta x^2}[2u_{I-1} - 3u_I + u_{II+1}]$. Similarly, we can take other different grid spacings.

We now consider the finite difference operator on uniform gird. Let $h_1 = h_2 = h$ as shown in Figure 10. Then we can obviously see that the classical-central finite difference approximation is $(L_h u)_I = \frac{1}{h^2}[u_{I-1} + u_{I+1} - 2u_I]$.

4.2 VCDLO in 2-D

We consider the finite difference approximation for the Laplacian $((Lu) = u_{xx})$ on a two dimensional non-uniform grid.

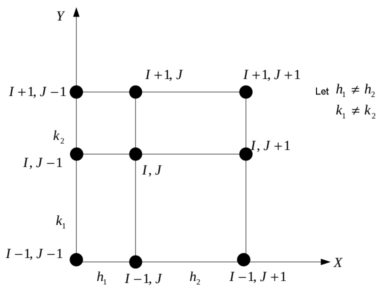

Figure 11. Non-Uniform Grid in 2-D

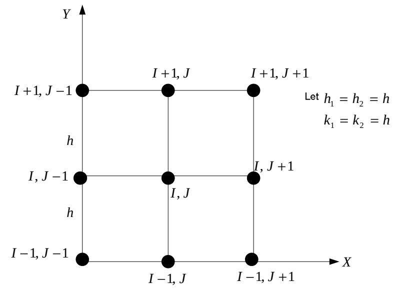

Figure 12. Uniform Grid in 2-D

Let us consider the grid spacings $h_1 = x_{I,J} - x_{I,J-1}$, $h_2 = x_{I,J+1} - x_{I,J}$, $k_1 = y_{I,J} - y_{I-1,J}$, $k_2 = y_{I+1,J} - y_{I,J}$. In 2D, the discrete Laplacian operator for node (I,J) can be written as

$$(L_h u)_{I,J} = \frac{1}{l_I}\left[\sum_{J=1}^{2}\alpha_{IJ}u_{IJ} - \alpha_I u_{IJ}\right] + \frac{1}{l_J}\left[\sum_{I=1}^{2}\alpha_{IJ}u_{IJ} - \alpha_J u_{IJ}\right]. \quad (8)$$

In this case, $l_J = \dfrac{k_1 + k_2}{2}$, $\alpha_{IJ} = k_I, I = 1,2$. Then we have

$$(L_h u)_{I,J} = \frac{2}{h_1 + h_2}\left[\frac{1}{h_1}u_{I,J-1} + \frac{1}{h_2}u_{I,J+1} - \left(\frac{1}{h_1} + \frac{1}{h_2}\right)u_{I,J}\right] + \frac{2}{k_1 + k_2}\left[\frac{1}{k_1}u_{I+1,J} + \frac{1}{k_2}u_{I+1,J} - \left(\frac{1}{k_1} + \frac{1}{k_2}\right)u_{I,J}\right]$$

and hence becomes,

$$(L_h u)_{I,J} = \frac{2}{h_1 h_2 (h_1 + h_2)}\left[h_2 u_{I,J-1} + h_1 u_{I,J+1}\right] + \frac{2}{k_1 k_2 (k_1 + k_2)}\left[k_2 u_{I-1,J} + k_1 u_{I+1,J}\right] - \left[\frac{(h_1 + h_2)}{h_1 h_2 (h_1 + h_2)} + \frac{(k_1 + k_2)}{k_1 k_2 (k_1 + k_2)}\right]2u_{I,J}. \quad (9)$$

Here, we take an example as in previous case. If we take $h_1 = \dfrac{3\Delta x}{4}$, $h_2 = \dfrac{3\Delta x}{2}$, $k_1 = \dfrac{\Delta y}{4}$,

$k_2 = \dfrac{3\Delta y}{4}$, then we get

$(L_h u)_{I,J} = \dfrac{16}{27\Delta x^2}\left(2u_{I,J-1} + u_{I,J+1}\right) + \dfrac{8}{3\Delta y^2}\left(3u_{I-1,J} + u_{I+1,J}\right) - \left(\dfrac{8}{9\Delta x^2} + \dfrac{16}{3\Delta y^2}\right)2u_{I,J}$. Then we get classical-central finite difference approximation for 2D on uniform-grid,

$$(L_h u)_{I,J} = \frac{1}{h^2}\left[u_{I,J-1} + u_{I,J+1} + u_{I-1,J} + u_{I+1,J} - 4u_{I,J}\right]. \quad (10)$$

4.3 VCDLO in 3-D

Let us consider the grid spacings,
$h_1 = x_{I,J,K} - x_{I-1,J,K}$, $h_2 = x_{I+1,J,K} - x_{I,J,K}$, $k_1 = y_{I,J,K} - y_{I,J-1,K}$, $k_2 = y_{I,J+1,K} - y_{I,J,K}$, $p_1 = z_{I,J,K} -$

$z_{I,J,K-1}$, $p_2 = z_{I,J,K+1} - z_{I,J,K}$. In 3-D, the discrete Laplacian operator for node (I,J,K) can be written as

$$(L_h u)_{I,J,K} = \frac{1}{l_I}\left[\sum_{J=1}^{2}\alpha_{IJK}u_{IJK} - \alpha_I u_{IJK}\right] + \frac{1}{l_J}\left[\sum_{I=1}^{2}\alpha_{IJK} - \alpha_J u_{IJK}\right] + \frac{1}{l_k}\left[\sum_{k=1}^{2}\alpha_{IJK}u_{IJK} - \alpha_K u_{IJK}\right]. \quad (11)$$

In this case, $l_k = \dfrac{p_1 + p_2}{2}$, $\alpha_{IJK} = p_I, K = 1,2$ and hence, we get

$$(L_h u)_{I,J,K} = \frac{2}{h_1 h_2 (h_1 + h_2)}\left[h_2 u_{I+1,J,K} + h_1 u_{I-1,J,K}\right] + \frac{2}{k_1 k_2 (k_1 + k_2)}\left[k_2 u_{I,J+1,K} + k_1 u_{I,J-1,K}\right]$$

$$+\frac{2}{p_1 p_2 (p_1+p_2)}\left[p_2 u_{I,J,K+1}+p_1 u_{I,J,K-1}\right]-\left[\frac{(h_1+h_2)}{h_1 h_2 (h_1+h_2)}+\frac{(k_1+k_2)}{k_1 k_2 (k_1+k_2)}+\frac{(p_1+p_2)}{p_1 p_2 (p_1+p_2)}\right]3u_{I,J,K} \qquad (12)$$

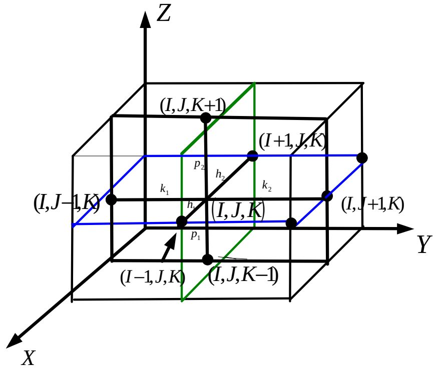

Figure 13. Non-Uniform Grid in 3-D

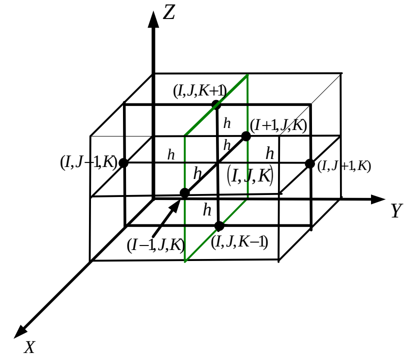

Figure 14. Uniform Grid in 3-D

In this case, the grid spacings are shown in Fig.14. By following the same idea as in previous cases, we can prove the classical-central finite difference approximation for 3-D on uniform grid, which is

$$(L_h u)_{I,J,K}=\frac{1}{h^2}\left[u_{I-1,J,K}+u_{I+1,J,K}+u_{I,J-1,K}+u_{I,J+1,K}+u_{I,J,K-1}+u_{I,J,K+1}-6u_{I,J,K}\right] \qquad (13)$$

5. CONCLUSION

The interpolants used to construct the trial and test functions are known as natural neighbor (n-n) interpolants. These interpolants are based on the Dirichlet or Voronoi tessellation of a set (N) of distinct nodes in the plane. The Voronoi tessellation is a unique and geometrically fundamental construct that defines a set of nodes and points. We can also see that VCDDO constructed on Voronoi diagram convert to VCDLO if $\kappa_{IJ}=1$. DLO can be reduced to classical central finite difference in 1D as well as 2D and 3D. VCDLO considered on classical central finite difference operator with four nodes on uniform grids in 2D can be obviously observed that it is equivalent to VCDDO if we choose at least four nodes and similarly in 3D.

6. ACKNOWLEDGEMENT

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