Numerical Analysis for One Dimensional Electromagnetic Problem

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ABSTRACT: In this paper, one dimensional electromagnetic problem is addressed with the help of Physical Spline Finite Element Method(PSFEM) and Analytical Method. In transmission line problems, the voltage distribution from the source can be calculated by using Physical Spline Finite Element Method. In this method, physical differential equations are incorporated into interpolation of basic elements in Finite Element Method. Moreover, Analytical Method is also used for voltage distribution. As a result, the accuracy of the Physical Spline Finite Element Method shows the great improvement of the solution.

1. INTRODUCTION

In electromagnetic and microwave engineering, electromagnetic waves are important both theoretically and practically. In applications, electromagnetic waves are basic components in microwave and optical networks. Various techniques have been published, Finite Difference (FD), Finite Difference Time-Domain (FDTD) and Finite Element Method (FEM). The node-based finite element method has been successfully applied to the electromagnetic problems since 1960s (Zhou, 2002). In research, one of the trends is to incorporate more efficient mathematical solvers into finite element method of large scale problems (Zhou, 2003). It has been used to solve mechanical problems, electro-magnetic problems and many other mathematical and physical problems.

Another interpolation technique, spline has been used for a long time inhomogeneous problems. Applications of spline functions in solving differential (linear and nonlinear) and integral equations are mostly limited to theoretical investigations. Very few applications in electrical engineering have been found. On the other hand, all the existing interpolations are inherently general mathematical tools. In this paper, a novel physical spline expansion is introduced. This idea is motivated by the following intuition: if we can incorporate the properties of the physical equations into the corresponding expansions, we can solve these physical problems much more efficiently and accurately. The only requirement on the expansion is that the functional must be differentiable with respect to the expansion coefficients. The physical spline expansion is based on the well-known cubic spline interpolation that is used widely in data processing, curve fitting and computer graphics (Zhou, 2002). However, traditional cubic spline is not convenient for finite element method implementation. Incorporation of the differential equations changes the situation dramatically. Physical spline finite element method which has been successfully developed and applied to one dimensional electromagnetic problem.

In this paper, only one-dimensional (1D) problem is discussed to emphasize the idea, procedure and power of the method without being distracted. Although 1D problems themselves are not very useful in general and analytical solutions exist in most cases, numerical solutions are preferred in some applications (W.H, 1992). This paper is organized as follows. Section 1 presents the general one dimensional electromagnetic problems. Section 2 constructs the physical spline interpolation and first order lagrange interpolation. Section 3 describes the physical spline finite element method that is used to the electromagnetic problems. Numerical example is presented in the section 4 to show the effectiveness of the physical spline finite element method.

2. MATHEMATICAL MODLE FOR 1-D ELECTROMAGNETIC PROBLEM

Most linear one dimensional electromagnetic problems can be recognized as special case of one dimensional sturm-liouville differential equation

$$-\frac{d}{dx}\left(p(x)\right)\frac{dU}{dx}+q(x)U(x)=f(x),0\leq x\leq L. \tag{2.1}$$

where, p(x) and q(x) are known functions and U(x) is unknown function (voltage distribution). We just treat equation (2.1) as a mathematical equation. Boundary conditions can be generalized as

$$U=V_0 \quad \left(x=0\ or\ x=L\right) \tag{2.2}$$

$$\frac{\partial U}{\partial x}+\alpha U=\beta, \quad \left(x=0\ or\ x=L\right). \tag{2.3}$$

Equation (2.1) claims that the second derivatives of U must be continuous in uniform regions, i.e. U belong to the smoothness class C_2 [0,1]. At an interface between different uniform regions, equation (2.1) holds on both sides. Equation (2.2) is essential boundary condition and equation (2.3) is generalized boundary condition.

3. PHYSICAL SPLINE EXPANSION

3.1. Cubic Spline Interpolation

Suppose that $\{(x_k, y_k)\}$ k = 0 ,…, N are N+1 points, where $a = x_0 < x_1 < .. < x_N = b$. The function S(x) is called a cubic spline function if there exist N cubic polynomial $S_k(x)$ with coefficients $S_{k,0}$, $S_{k,1}$,$S_{k,2}$, $S_{k,3}$ and that satisfy the following properties.

(1) $S(x) = S_k(x) = S_{k,0} + S_{k,1}(x-x_k) + S_{k,2}(x-x_k)^2 + S_{k,3}(x-x_k)^3$
for x belong to $[x_k, x_{k+1}]$ and k = 0,…,N-1.
(2) $S(x_k) = y_k$ k = 0,…,N.
(3) $S_k(x_{k+1}) = S_{k+1}(x_{k+1})$ k = 0,…,N-2.
(4) $S'_k(x_{k+1}) = S'_{k+1}(x_{k+1})$ k = 0,…,N-2.
(5) $S''_k(x_{k+1}) = S''_{k+1}(x_{k+1})$ k = 0,…,N-2.

(1) states that S(x) consists of piecewise cubics. (2) states that the piecewise cubics interpolation in the given set of data points. (3) and (4) require that the piecewise cubic represent a smooth continuous function. (5) means second derivative of function is also continuous.
Since U(x) is piecewise cubic and its piecewise linear on $[x_0,x_N]$.
The linear lagrange interpolation formula is

$$y=Ay_j+By_{j+1} \tag{3.1}$$

where,

$$A=\frac{x_{j+1}-x}{x_{j+1}-x_j}, \qquad B=1-A. \tag{3.2}$$

Cubic spline may be the best starting point since its first order derivatives are smooth and the second order derivatives are continuous. This property is good for uniform regions but too much for discontinuous dielectrics. It is also not easy to implement.

3.2. Cubic Spline Expansion

U(x) is (voltage distribution) the behavior of the function (W.H. and S.A., Tewkdsky, 1992).
Cubic spline function within an element (x_1^e, x_2^e) is

$$U^e(x)=\sum_{i=1}^{2}\left[N_i^e(x)U_i^e+M_i^e(x)(U_i^e)''\right] \tag{3.3}$$

where,

$$N_1^e(x) = \frac{x_2^e - x}{x_2^e - x_1^e} \quad , \quad N_2^e(x) = \frac{x - x_1^e}{x_2^e - x_1^e} = 1 - N_1^e(x) \qquad (3.4)$$

$$M_1^e(x) = \frac{1}{6}\left[(N_1^e)^3 - N_1^e\right](x_2^e - x_1^e) \ , \ M_2^e(x) = \frac{1}{6}\left[(N_2^e)^3 - N_2^e\right](x_2^e - x_1^e). \qquad (3.5)$$

Equation (3.3) is called the cubic spline interpolation equation. Obviously, $0 \le |N^e_i| \le 1$, $M^e_i \le 0$.

3.3. Physical Spline Interpolation

In electromagnetic, p(x) and q(x) are associated with the characteristics of dielectrics. It is reasonable to make p(x) = p^e and q(x) = q^e are constants within an element.
From equation (2.1) we get,

$$U_i^{e''} = -\frac{1}{p^e} f(x_i^e) + \frac{q^e}{p^e} U_i^e , (i = 1,2) \qquad (3.6)$$

by substituting equation (3.3) in equation (3.6),

$$U^e(x) = \sum_{i=1}^{2}\left[N_i^e(x) + \frac{q^e}{p^e} M_i^e(x)\right] U_i^e - \frac{1}{p^e} f(x_i^e) M_i^e(x). \qquad (3.7)$$

Equation (3.7) is a combination of cubic spline and physical equation that is called physical spline interpolation equation.

4. IMPLEMENTATION OF FINITE ELEMENT METHOD

4.1. Finite Element Method Approach

Finite element method has been a very powerful tool and can be used for the accurate solution of complex engineering problems. In finite element method, the differential equation described the transmission line are not tackled directly. Instead the method takes advantage of the equivalent physical principle that the voltage distribution along the line will always so adjust itself as to minimise power loss. While such redistribution is sometimes difficult to express mathematically if an exact solution is required, it is not hard to formulate in an approximate sense (Sil.,1996).
The following procedures are finite element method approach

(1) Express the power W lost in the line in terms of the voltage distribution v(x)

W = W [v(x)] .

(2) Subdivide the domain of interest (the entire transmission line) into K finite sections or elements (Kth elements).
(3) Approximate the voltage v(x) along the line using a separate approximating expression in each element of the form

$$v(x) = \sum_{i=1}^{M} v_i f_i$$

where the f_i(x) are some convenient set of know functions. These expressions will necessily include M constant but as yet unknow coefficient v_i on each element for the voltage along the line is not yet known.
(4) Express power in each element in terms of the approximating functions $f_i(x)$ and their M undetermined coefficients v_i. Because the functions $f_i(x)$ are chosen in advance and therefore known the power becomes a function of the coefficient only

W=W($v_1,\ldots,v_M$).

(5) Introduce constraints on the MK coefficients so as to ensure the voltage is continuous from element to element.

Thus constrained, the ensemble of all elements will possess some N degree of freedom $N \le MK$.
(6) Minise the power by varing each coefficient v_i in turn, subject to the constraint that voltage along the line must in a continuous fashion

$$\frac{\partial W}{\partial v_i} = 0, i = 1,..., N .$$

This minimisation determines the coefficient and thereby produces an approximate expression for the voltage along the line.

4.2. Physical Spline Finite Element Method

In this method, physical differential equations are incorporated into interpolations of basic elements in finite element method. This is named physical spline finite element method. Theoretically, the physical spline interpolation introduces many new features. First, physical equations can be used in the interpolations to make the interpolations problem-associated. Then, the combination of cubic spline and physical equation is physical spline equation and its corresponding finite element implementation is called physical spline finite element method.

5. NUMERICAL EXAMPLE

5.1. A Simple Lossy Direct Current Transmission Line

A transmission line is a system of conductors connecting one point to another and along which electromagnetic energy can be sent. For examples, telephone lines and power transmission lines.

Figure 1. Power Transmission Line.

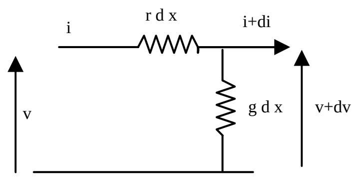


Figure 2. Equivalent Circuit.

We consider a direct current power transmission line length (L) is 2. In this situation, it has only very low frequency but the transmission line is resistive. So there will be some longitudinal voltage drop as well as current through the pipe. Assume that, a resistivity value (r) is 2, conductivity per unit length (g) is 0.5 and the voltage value at the length is V_0. Now, we determine the distribution of the voltage from the source.

5.2. Analytical Method

In a short length dx of the line in figure 2 is resistive so longitudinal voltage drop as well as current (Sil., 1996). While current di flow, represented by the shunt conductance g dx

$$dv = -irdx \tag{5.1}$$

$$di = -(v + dv)\, gdx \tag{5.2}$$

Then, discarding second-order term from equation (5.1) and (5.2)

$$\frac{dv}{dx} = -ri \tag{5.3}$$

$$\frac{di}{dx} = -gv \tag{5.4}$$

differentiating with respect to x, we get voltage and current equations

$$\frac{d^2v}{dx^2} = r\frac{di}{dx} \tag{5.5}$$

$$\frac{d^2i}{dx^2} = g\frac{dv}{dx} \tag{5.6}$$

the governing transmission line equation for voltage is

$$\frac{d^2v}{dx^2} - rgv = 0 \tag{5.7}$$

by using boundary conditions

$$v(x = l) = V_0 \tag{5.8}$$

$$\frac{dv}{dx}_{(x=0)} = 0.$$

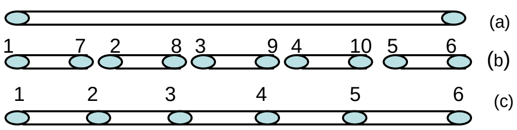
We get the analytical solution of voltage distribution in transmission line

$$v = V_0 \frac{e^{\sqrt{(rg)}x} + e^{-\sqrt{(rg)}x}}{e^{-\sqrt{rgl}} + e^{\sqrt{rgl}}}. \tag{5.9}$$

5.3. Physical Spline Finite Element Method (PSFEM)

In transmission line problem, the voltage distribution from the source can be calculated the following procedures.

- **Subdivide into the Five Elements Model**



(a) Original Transmission Line Problem.
(b) Elements Numbering and Nodes Identification with Element Considered Separately.
(c) Numbering of Nodes after Connection.

- **Procedures of Finite Element Method**

- Single element
- K segments or finite elements
- Joining the elements

5.4. Physical Spline Finite Element Method

Consider a short section dx of the line as in figure 2. The power entering this section at its left end is given by

$$W_{in} = v_i \tag{5.10}$$

while the power leaving on the right is

$$W_{out} = (v + dv)(i + di). \tag{5.11}$$

The difference between the incoming and outgoing amounts must represent the power lost in the section dx. Neglecting the second-order term dv di, this difference is

$$dW = vdi + idv \tag{5.12}$$

The power per unit length then takes the form is

$$\frac{dW}{dx} = - gv^2 - \frac{1}{r}\left(\frac{dv}{dx}\right)^2 \tag{5.13}$$

and the total power for the whole line is

$$W = -\int_0^L \left[gv^2 + \frac{1}{r}\left(\frac{dv}{dx}\right)^2 \right] dx. \tag{5.14}$$

5.5. Piecewise Approximation of Voltage for the Transmission Line

We consider voltage varies linearly with distance x within any one element

$$v = \frac{x_{(k)r} - x}{x_{(k)r} - x_{(k)l}} v_{(k)} l + \frac{x - x_{(k)l}}{x_{(k)r} - x_{(k)l}} v_{(k)} r$$

$$v = \alpha_l(x) v_l + \alpha_r(x) v_r \tag{5.15}$$

where,

$$\alpha_l(x) = \frac{x_r - x}{x_r - x_l}, \qquad \alpha_r(x) = \frac{x - x_l}{x_r - x_l}. \tag{5.16}$$

The sum of power in the Kth element is given by

$$W = \sum_{k=1}^{K} W_k \tag{5.17}$$

where,

$$W_k = -\int_{x(k)l}^{x(k)r} \left[gv^2 + \frac{1}{r}\left(\frac{dv}{dx}\right)^2 \right] dx. \tag{5.18}$$

Substituting equation (5.15) in equation (5.18), the power in the Kth element is

$$W = -\frac{1}{r_k} \int_{x(k)l}^{x(k)r} \left[v_1 \frac{d\alpha_l}{dx} + v_r \frac{d\alpha_r}{dx} \right]^2 dx - g_k \int_{x(k)l}^{x(k)r} \left[v_l \alpha_l + v_r \alpha_r \right]^2 . dx. \tag{5.19}$$

The above equation can be written in matrix quadratic form as follows,

$$W_k = -\begin{bmatrix} v_l & v_r \end{bmatrix} \left[\frac{1}{r_k} S' + g_k T' \right] \begin{bmatrix} v_l \\ v_r \end{bmatrix} \tag{5.20}$$

here, S' and T' are 2x2 matrices with entries

$$S_{ij} = \int_{x_l}^{x_r} \frac{d\alpha_i}{dx} \frac{d\alpha_j}{dx} dx, \qquad T_{ij} = \int_{x_l}^{x_r} \alpha_i \alpha_j dx \tag{5.21}$$

More compactly define for the power in each element,

$$W_k = V_{(k)}{}^T M V_{(k)} \tag{5.22}$$

where,

$$M = \frac{1}{r_k} S' + g_k T'$$

5.6. Transformation of normalized coordinate

Let $L_k = x_{(k)r} - x_{(k)l}$ denotes the length of Kth element.

The normalized coordinate $\varsigma = \dfrac{x - x_l}{L_k}$ defined within the Kth element.

The two approximation functions can be written as

$$\alpha_l(\varsigma) = 1 - \varsigma$$
$$\alpha_r(\varsigma) = \varsigma, \qquad \varsigma \text{ ranges over } 0 \le \varsigma \le 1.$$

The approximate expression for voltage is $v = v_l \alpha_l(\varsigma) + v_l \alpha_r(\varsigma)$.

We get,

$$\frac{dv}{dx} = \frac{dv}{d\varsigma}\frac{d\varsigma}{dx} = \frac{1}{L_k}(v_r - v_l).$$

5.7. Total Power for Typical Element

$$S' = \frac{1}{L_k} S'', T' = L_k T''$$

where,

S'' = normalised matrix
T'' = normalised matrix

$$S'' = \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}, \; T'' = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{5.23}$$

by substituting in equation (5.20)

$$W_k = -\begin{bmatrix} v_l & v_r \end{bmatrix}\left[\frac{1}{r_k L_k}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} + \frac{g_k L_k}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} v_l \\ v_r \end{bmatrix}\right]. \tag{5.24}$$

Equation (5.24) is total power equation in typical element.

5.7. Joining the Elements

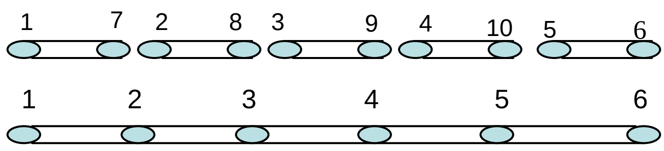


Figure 3. Joining Five Elements in Transmission Line.

To joint five elements, we use voltage connection equation is

$$v_{dis} = C v_{con} \tag{5.25}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \end{bmatrix}_{dis} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}_{con} \tag{5.26}$$

where,

v_{dis} = disconnected voltage

v_{con} = connected voltage

By using equation (5.25) and substituted in (5.27). We get the total power for five elements

$$W = -v_{con}{}^{T} (\frac{1}{r} C^T SC + gC^T TC) v_{con} \;\; or \;\; W = -v_{con}{}^{T} M v_{con} \tag{5.27}$$

$$W = -v_{con}{}^{T} C^T M_{dis} C\, v_{con}. \tag{5.28}$$

5.8. Power Minisation in Transmission Line

$$\frac{\partial W}{\partial v_i} = 0, i = 1, \ldots, N-1. \tag{5.29}$$

where,

W = total power

$\frac{\partial}{\partial v_i}$ = power minised operator

For five elements, N = 6, so the differentiation results in

$$\begin{bmatrix} M_{11} & M_{12} & & & & \\ M_{21} & M_{22} & & & & \\ & & & & & \\ & & & & & \\ & & & & M_{55} & M_{56} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = 0 \quad . \tag{5.30}$$

Simultaneous equation (5.30) can be solved and we obtained the exact solution for the voltage distribution

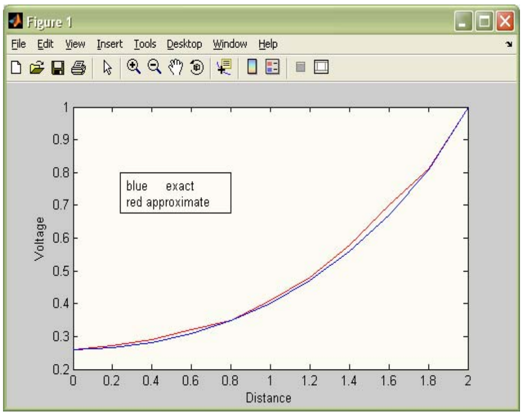


Figure 4. Voltage Distribution of Uniform Direct Current Transmission Line.

6. CONCLUSION

In this paper, one dimensional electromagnetic problem is successfully studied by using numerical analysis; physical spline finite element method and analytical method. We applied these methods are extended to 1D electromagnetic problem, namely transmission line problem. In PSFEM, physical differential equations are first incorporated into interpolations of basic elements in finite element method. The corresponding method is also used. We first construct five elements model and we can directly calculate voltage distribution from the source by using these methods without power loss along the transmission line. Physical Spline Finite Element Method(PSFEM) shows the great improvement of convergence and accuracy of voltage distribution.

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