Numerical Solution for Electrical Impedance Tomography Problem by D-bar Method

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ABSTRACT: In electrical impedance tomography (EIT) problem, an approximate for the internal resistivity distribution is computed based on the knowledge of the injected currents and measured voltages on the surface of the body. In this research, theoretical and numerical studies of the inverse problem of EIT which seeks the electrical conductivity and permittivity inside a body, given simultaneous measurements of electric currents and potential at the boundary are reviewed. In particular, brief description of EIT problem and D-bar method are described. D-bar method is an important technique to approximate the internal resistivity of the body. Then, the governing equation for EIT problem is derived form Maxwell's equations to compute impedance inside the body. To illustrate imaging activity of EIT problem, the application of D-bar algorithm is included with the numerical examples.

1. INTRODUCTION

Electrical properties such as the electrical conductivity σ and the electric permittivity ε, determine the behavior of materials under the influence of external electric fields. Let us consider a bounded, simply connected set $\Omega \subset R^2$ and, at frequency ω, let γ be the complex admittivity function $\gamma(z,\omega) = \sigma(z) + i\omega\varepsilon(z)$ where $i = \sqrt{-1}$. In the static case, if $\omega = 0$, then $\gamma(z) = \sigma(z)$. The electrical impedance is the inverse of $\gamma(z)$ and it measures the ratio between the electric field and the electric current at location $z \in \Omega$. EIT is a new imaging technique. EIT is the inverse problem of determining the impedance in the interior of Ω, given simultaneous measurements of direct or alternating electric currents and voltages at the boundary ∂Ω. Due to this fact, EIT is an imaging tool with important applications in fields such as medicine, geophysics, environmental sciences and nondestructive testing of materials. This system uses these electrical measurements to reconstruct and display approximate pictures of the electric conductivity and permittivity inside the body. This process is called EIT. The term “impedance” comes from circuit theory: it is the ratio of the voltage across a circuit element to the current through the element. EIT may provide an inexpensive and portable method for besides imaging of cardiac activity and pulmonary perfusion. Examples of medical application of EIT are the detection of pulmonary emboli (Cheney M, 1999 and Borcea, 2002).

2. MATHEMATICAL MODEL FOR EIT PROBLEM

The 2-D EIT problem so-called inverse conductivity problem is to determine and reconstruct an unknown conductivity distribution in an open, bounded and smooth domain and assume that the conductivity $\gamma \in L^\infty(\Omega)$ satisfies $\gamma \geq c > 0$ for some constant c. In a typical medical EIT problem, current is applied to electrode attached to the skin, the resulting voltages at a point z are measured, and then, using a variety of inversion techniques, the conductivity distribution is recovered in the static case. More formally, a current pattern is applied to the surface of body Ω, and the resulting electrical potential $\phi(z)$ on $\partial\Omega$ is measured.

Let $u(z)$ denote the voltage at a point z inside the body. Let n denote the unit outward normal to the surface. The boundary measurements are modeled by the Dirichlet-to-Neumann (DN) map

$$\Lambda_\gamma \phi(z) = \gamma(z)\frac{\partial u(z)}{\partial n} = j(z) \; on\,\partial\Omega \tag{1}$$

where $j(z)$ is the injected current. We now construct the mathematical model for EIT problem. The voltage and conductivity are assumed to satisfy:

$$\begin{aligned} \nabla.[\gamma(z)\nabla u(z)] &= 0 && \text{in} \quad \Omega \\ u(z) &= \phi(z) && \text{on} \quad \partial\Omega \\ j(z) &= \Lambda_\gamma \phi(z) && \\ &= \gamma(z)\frac{\partial u(z)}{\partial n} && \text{on} \quad \Omega \\ \gamma &= 1 && \text{near } \partial\Omega.. \end{aligned} \tag{2}$$

The problem (2) is the main problem for EIT problem and the first equation in it is called the conductivity equation.

3. D-BAR METHOD (Procedure to Solve Main Problem)

The following are the procedure to solve the main problem (2) which is known as D-bar method. D-bar method was detail expressed in (Nachmann, 1996, Muller and Siltanen, 2000). The terminology D-bar comes from a phonetic way of writing $\bar{\partial}$, the partial derivative with respect to the conjugate of a complex variable. This differential operator is defined for a complex variable $z = x + iy \in C$ at a point $z = (x, y) \in R^2$ by

$$\bar{\partial} = \bar{\partial}_z = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad . \tag{3}$$

The D-bar method is used to find and reconstruct the electrical conductivity $\gamma(z)$ inside a body Ω from the DN map Λ_γ. The reconstruction of $\gamma(z)$ can be divided into two steps. It is based on the used of intermediate object called the scattering transform $t(k)$. The $t(k)$ can be expressed as: $\Lambda_\gamma \to t \to \gamma$ where the intermediate object $t(k): C \to C$ is called the scattering transform, and which is not directly measurable in experiments.

Step 1: From Λ_γ to t ($\Lambda_\gamma \to t$)

The first step is to transform the conductivity equation in problem (2) to the Schrödinger equation mentioned in problem (5) via the change of variables:

$$\begin{aligned} \psi &= \psi(z,\varsigma) \equiv \gamma^{1/2} u, \\ q &= q(z) \equiv \gamma^{-1/2}\Delta\gamma^{1/2} \end{aligned} \tag{4}$$

where $\varsigma = k(1,i)$, $k = k_1 + ik_2$, $k \in C \setminus 0$ and $q(z)$ is the scattering potential. By using (4), we can construct the following problem (5) with the unknown variable $\psi(z,\varsigma)$

$$-\Delta\psi + q\psi = 0 \qquad in \quad \Omega$$
$$\Lambda_\gamma \psi = \partial\psi \big/ \partial n \qquad on \quad \partial\Omega \tag{5}$$

and q = 0 near $\partial\Omega$. Look for solution ψ on all of R^2 with q = 0 outside $\partial\Omega$ that satisfy $\psi \approx \exp(i\varsigma.z)\ as|z| \to \infty$, where $\varsigma.\varsigma = 0$.

By taking

$$\psi = \psi(z,\varsigma) = \exp(i\varsigma.z)\mu(z,\varsigma)$$
$$\mu(z,\varsigma) = 1 + \psi(z,\varsigma). \tag{6}$$

where $\mu(z,k)$ is the magnetic permeability and $\mu \to 1$ as $|z| \to \infty$.

Step 2: From t to Λ_γ ($t \to \gamma$)

D-bar equation and scattering transform

By taking the differentiation of $\psi(z,\varsigma)$ with respect to k and applying the d-bar operator (3) to that equation yields **D-bar equation** (Nachman, 1996, Borcea, 2002) and it can be expressed with respect to the unimodular function $e_k(z)$ and the **scattering transform** $t(k)$ as follows:

$$\frac{\partial\mu}{\partial\bar{k}} = \frac{1}{4\pi\bar{k}} t(k) e_{-k}(z)\overline{\mu}(z,k) \quad k \in C \setminus 0,\ z \in R^2. \tag{7}$$

The functions $e_k(z)$ and $t(k)$ in D-bar equation (7) are defined by $e_k(z) = e^{2i(k_1 x - k_2 y)}$ and

$$t(k) = \int_{\partial\Omega} e_k(z) q(z) \mu(z,k)\, dx dy,\ \ z = x + iy$$

respectively. The solution $\mu(z,k)$ of the D-bar equation (7) can be written as

$$\mu(z,k) = 1 + \frac{1}{(2\pi)^2}\int_{R^2} \frac{t(k)}{(s-k)\bar{k}} e_{-z}(k)\overline{\mu(z,k)}\, dk_1 dk_2 \quad . \tag{8}$$

Taking the small k limit of $\mu(z,k)$ gives the conductivity $\gamma(z)$ directly at each point z in Ω,

$$\lim_{k\to 0} \mu(z,k) = \gamma^{1/2}(z), \quad z \in \Omega. \tag{9}$$

Here we note that permeability $\mu(z,k)$ is the square root of $\gamma(z,k)$. Since both $\gamma^{1/2}$ and $\mu \to 1$ at ∞, they are identical. This procedure is called D-bar method.

4. Classification of Scattering Transform

4.1. The boundary scattering transform

D-bar method is based on the evaluation of the scattering transform $t(k)$. If $\psi(.,k)|_{\partial\Omega}$ has been determined, and then $t(k)$ can be recovered from the formula

$$t(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{z}} (\Lambda_\gamma - \Lambda_1)\psi(.,k)\, d\gamma(z), \quad k \in C, z = x + iy. \tag{10}$$

where Λ_1 denote the DN map corresponding to the homogeneous conductivity 1. Then γ can be recovered by solving a D-bar equation containing $t(k)$. The scattering transform contains enough information to recover γ. The functions $\psi(.,k)$ in (10) are traces of certain exponentially growing solution to first equation of problem (2), i.e., solutions that behave like e^{ikz} asymptotically as either $|z|$ or $|k|$ tend to infinity.

D-bar method using t^B

The approximation for $t(k)$ from the formula $t^B(k)=\int_{\partial\Omega} e^{i\bar{k}\bar{z}}(\Lambda_\gamma-\Lambda_1)\psi^B(.,k)d\gamma$. Since t^B also blow up as $|k|\to\infty$ we cut-off as before and define $t_R^B(k)=\begin{cases} t^B(k), & |k|<R \\ t^B(k)\chi_R(|k|), & |k|\geq R. \end{cases}$ Replacing the exact scattering transform $t(k)$ by t^B in the D-bar equation (8) and then gives the solution $\mu_R^B(z,s)=1+\frac{1}{(2\pi)^2}\int_{R^2}\frac{t_R^B(k)}{(s-k)\bar{k}}e_{-x}(k)\frac{t_R^{\exp}(k)}{|k|^2}dk_1dk_2$. This equation gives us the reconstruction $\gamma_R^B(z)=(\mu_R^B(z,0))^2$.

4.2. The Approximate Scattering Transform

The approximation to the scattering transform $t(k)$. The boundary integral expression is introduced

$$t^{\exp}(k)=\int_{\partial\Omega} e^{i\bar{k}\bar{z}}(\Lambda_\gamma-\Lambda_1)e^{ikz}d\gamma(z). \tag{11}$$

As in (Calderon, 1980), we expand e^{ikz} in a Fourier series with $z = e^{i\theta}$ to obtain

$$e^{ikz} = \sum_{n=-\infty}^{\infty} a_n(k)e^{in\theta} \text{ with } a_n(k)=\begin{cases} \frac{(ik)^n}{n!}, & n\geq 0 \\ 0, & n<0 \end{cases}.$$

D-bar method using t^{exp}

For $c^{-1}<\gamma(x)<c, \quad c>0,\ x\in\Omega$, The approximation $t^{\exp}$ is defined by substituting $e^{ikz}|_{\partial\Omega}$ for $\psi|_{\partial\Omega}$ in equation (10) and computing $t^{\exp}(k)=\int_{\partial\Omega} e^{i\bar{k}\bar{z}}(\Lambda_\gamma-\Lambda_1)e^{ikz}d\gamma(z)$. Since $\chi_R(|k|)$ grows like $|k|^{1/2}$ as $|k|\to\infty$ (K.Kunden, 2006), the approximate scattering transform is restricted to a disk of radius R in the complex plane $t_R^{\exp}(k)=\begin{cases} t^{\exp}(k), & |k|<R \\ t^{\exp}(k)\chi_R(|k|), & |k|\geq R. \end{cases}$ Here $\chi_R(|k|)$ is either a hard or soft cut-off function near $|k|=R$. Then t is replaced by $t^{\exp}$ in equation (11) and we solve this equation and compute the reconstruction $\gamma_R^{\exp}(z)=(\mu_R^{\exp}(z,0))^2$.

4.3. The differencing scattering transform

Next, we compute the differencing scattering transform t^{diff} which is a modification of the approximate scattering transform in **(**Isaacson, 2004). It tailored to form difference images. Here we use the complex notation $\exp(ikz)=\exp(i(k_1+ik_2)(x+iy))$. Since evaluation of the righthand side of (11) is stable only for $|k| \le R$ where the radius R depends on the noise level, $t^{\exp}$ must be truncated. For $\gamma \ge c > 0$.

Set

$$t_R^{\exp}(k;\gamma) = \begin{cases} t^{\exp}(k;\gamma) & for\ |k| \le R \\ 0 & for\ |k| \rangle R \end{cases} \tag{12}$$

and hence

$$\begin{aligned} t_R^{dif} &= t_R^{\exp}(k;\gamma_1) - t_R^{\exp}(k;\gamma_2) \\ &= \int_{\partial\Omega} e^{i\bar{k}\bar{z}}\left(\Lambda_{\gamma_1} - \Lambda_1\right)e^{ikz}\,d\gamma(z) - \int_{\partial\Omega} e^{i\bar{k}\bar{z}}\left(\Lambda_{\gamma_2} - \Lambda_1\right)e^{ikz}\,d\gamma(z) \\ &= \int_{\partial\Omega} e^{i\bar{k}\bar{z}}\left(\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\right)e^{ikz}\,d\gamma(z). \end{aligned}$$

The series formulation for $t_R^{\exp}$ derived in (Isaacson, 2004) is used in the actual implementation. The D-bar method changes conductivities from EIT problem by using the differencing scattering transform and can get difference images from the reference frame.

Table 1. Piecewise Constant Conductivities by D-bar Method

Case	Conductivity	Scattering Transform	Truncated Radius	Recovered Conductivity	Remark
1.	$c^{-1} < \gamma < c$	$t(k)$	$\|k\| < R$	$\gamma = (\mu(x,0))^2$	Conductivity and resistivity for all cases are discontinuous
2.	$c^{-1} < \gamma < c$	$t_R^{\exp}(k)$	$\|k\| < R$	$\gamma_R^{\exp} = (\mu_R^{\exp}(x,0))^2$	
3.	$c^{-1} < \gamma < c$	$t_R^B(k)$	$\|k\| < R$	$\gamma_R^B = (\mu_R^B(x,0))^2$	

5. Discussion with Numerical Examples

In this section, we present reconstruction of high-contrast and low-constrast conductivities. We include two radially symmetric conductivities. We know that for a radial conductivity, the scattering transform $t(k)$ is rotationally invariant and real-valued (Nachman, 1996). Thus, it is enough to find $t(k)$ for a collection of positive real numbers instead of a 2D grid of k-values. Here we consider the domain interval $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$ to solve the forward problem of computing $\mu(z,k)$ from knowledge of γ. We consider an example by construction a building block for smooth conductivities. In this example, we consider radially symmetric low and high-contrast conductivities. Let $0 < d < 1$ and define a non-negative function $\psi_d \in C_0^\infty(R)$ by

$$\psi_d(t) = \begin{cases} e^{-\frac{2(d^2+t^2)}{(t+d)^2(t-d)^2}} & for\ -d \le t \le d \\ 0 & otherwise. \end{cases}$$

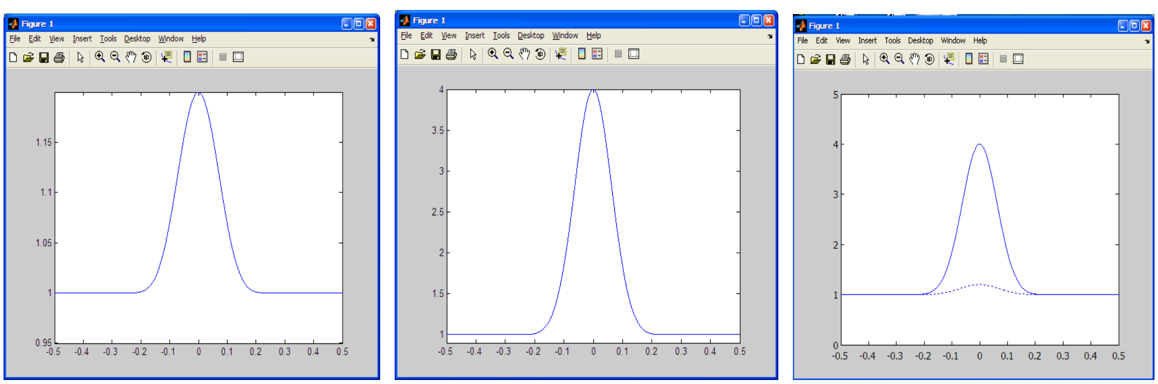
Define a radially symmetric function $\gamma \in C^\infty(R)$ by

$$\gamma(x) = (\alpha\psi_d(x) + 1)^2, \tag{13}$$

where α is called regularization parameter and $\alpha > -\exp(2/d^2)$ is a real constant. Various regularization parameters α give oscillatory images of γ and which give a γ that is too smooth. We let γ be given by (13) with $d = 1/2$, $\alpha = (\sqrt{6/5} - 1)e^8$ and $\alpha = e^8$ respectively. The conductivities are plotted in Figure 1 (in 2D) and Figure 2 (in 3D), have amplitudes 1.2 and 4, and are very near one for $|x| > 0.2$ (see Table 2).

Table 2. Amplitude of Conductivity in Low-Contrast and High-Contrast Media for $t \in [-\frac{1}{2}, \frac{1}{2}]$

Case	Conductivity $\gamma(x)$	Distance d	Regularization parameter α	Amplitude of $\gamma(x)$		Remark
				$\|x\| < 0.2$	$\|x\| > 0.2$	
1	low-conductivity	1/2	$(\sqrt{6/5} - 1)e^8$	1.2	$\cong 1$	accurate conductivity values
2	high-conductivity	1/2	e^8	4	$\cong 1$	Not accurate conductivity values



(a) (b) (c)

Figure 1. Amplitude of Conductivity in 2D (a) Low-Contrast Media (b) High-Contrast Media (c) The comparison for both Cases

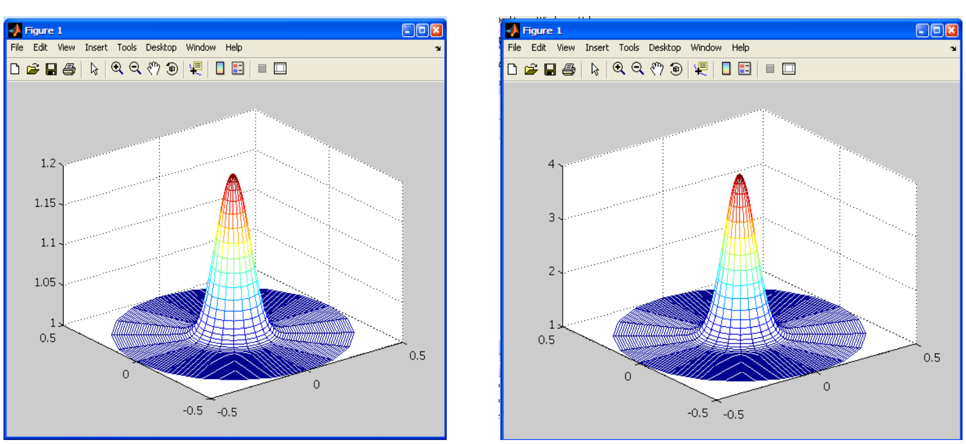


Figure 2. Amplitude of Conductivity in 3D (a) Low-Contrast Media (b) High-Contrast Media

If we consider γ with $d = \frac{1}{3}$ and $\frac{1}{4}$ in the example, then the conductivities are the same values as in the previous case, and very near one for $|x| > 0.1$ and $|x| > 0.05$ respectively (see Table 3 & 4). We obviously see that accuracy of $\gamma(x)$ depends on regularization of parameter α and the smaller the interval of the distance *d*, the more the accuracy of the conductivity $\gamma(x)$ values.

Table 3. Amplitude of Conductivity in Low-Contrast and High-Contrast Media for $t \in [-\frac{1}{3}, \frac{1}{3}]$

Case	Conductivity $\gamma(x)$	Distance d	Regularization parameter α	Amplitude of $\gamma(x)$		Remark
				$\lvert x\rvert < 0.1$	$\lvert x\rvert > 0.1$	
1	low-conductivity	1/3	$\left(\sqrt{6/5}-1\right)e^{18}$	1.2	$\cong 1$	accurate conductivity values
2	high-conductivity	1/3	e^{18}	4	$\cong 1$	Not accurate conductivity values

Table 4. Amplitude of Conductivity in Low-Contrast and High-Contrast Media for $t \in [-\frac{1}{4}, \frac{1}{4}]$

Case	Conductivity $\gamma(x)$	Distance d	Regularization parameter α	Amplitude of $\gamma(x)$		Remark
				$\lvert x\rvert < 0.05$	$\lvert x\rvert > 0.05$	
1	low-conductivity	1/4	$\left(\sqrt{6/5}-1\right)e^{32}$	1.2	$\cong 1$	accurate conductivity values
2	high-conductivity	1/4	e^{32}	4	$\cong 1$	Not accurate conductivity values

6. CONCLUSION

Electrical Impedance Tomography problem finds conductivity inside the materials by using the different types of scattering transform. Internal state of the materials demonstrates by using the differencing scattering transform and the conductivity of real data displays by using of approximate scattering transform. The D-bar method presented in this note can distinguish between different phases of the cardiac cycle. This method could be used to obtain useful images on measured data of the materials and displays to more general conductivities by using different types of scattering transform.

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