EXTENDED FINITE ELEMENT METHOD IN MODELING CRACKS

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ABSTRACT: To model a crack, the regular Finite Element Method (FEM) would have to exactly mesh the geometry of the crack and use a very fine mesh near the crack tip to capture the singular stress field. When the crack grows, cumbersome remeshing is required. To relieve the task in modeling cracks, several new finite element techniques are explored. Among them, it is seen that Extended Finite Element Method (XFEM) is very powerful technique for modeling of cracks and crack growth.

Indeed, XFEM is an enrichment technique in the standard finite element framework through a partition of unity method. XFEM model consists of a regular FEM which is independent of the crack geometry and a crack representation which is independent of the elements. In XFEM, special functions are added to the finite element approximation using the framework for partition of unity.

In this study, we analyse the determination and classification of the approximation of the discontinuous displacement field to model the several types of cracks according to the alignment within the mesh arbitrarily by using XFEM.

1. INTRODUCTION

In the Extended Finite Element Method, the emphasis has been on modeling discontinuities (such as cracks) with minimal enrichment (Belytschko, 1999). The modeling of moving discontinuities with the finite element method is cumbersome due to the need to update the mesh ‘topology’ to match the geometry of the discontinuity (Nicolas, 1999). The XFEM circumvents these problems by enriching a standard mesh-based approximation with additional discontinuous functions. The formed of the enriched approximation follows the partition of unity framework. The signed distance property of the level set is exploited to form the enrichment function (Stolarska, 2001). In XFEM, the standard displacement-based finite element approximation is enriched near a crack by incorporating both the discontinuous field and the singular asymptotic crack tip field.

The key idea of the XFEM is that part of the displacement field is approximated by a discontinuous displacement enrichment based on a local partition of unity (Belytschko, 1999). The displacement field is approximated by the sum of the regular displacement field, which is the displacement without any discontinuities, and the enrichment displacement field, which is the additional displacement that models the discontinuities. The method is applied to fracture mechanics, in which discontinuities are represented using a both a jump function and the asymptotic near-tip crack field can also be included to improve accuracy. XFEM has subsequently been extended too many applications: step function enrichment, crack growth with friction, arbitrary branched and interesting cracks, three dimensional crack propagation, material discontinuity problems etc.

2. MODELING OF CRACK

On the position of the crack we construct the mesh to investigate the formula of the approximation of the displacement field by XFEM. In crack modeling, different functions are used to enrich the displacement approximation in the interior of the crack and on the crack front. Therefore, one of the first tasks is to determine the finite elements that intersect the crack. In two-dimensions, cracks are represented by line segments and the crack front is a point. In three dimensions, a crack is represented by a polygon partition into triangles and the crack front consists of line segments. In all cases, the discontinuous approximation is constructed in terms of a signed distance functions, so level sets can be used to update the position of the discontinuities (Stolatska, 2001).

2.1. Signed Distance Function and Level Set Method(LSM)

The surfaces of discontinuity are defined by signed distance functions. This description is not necessary for the application of these discontinuous approximations, but they are very appealing because the methodology of level sets can then be applied to update these surfaces for moving discontinuities. The distance d from a point x to an interface Γ is defined as $d = \|x - x_\Gamma\|$ where x_Γ is the normal projection of x on Γ. The signed distance function $\phi(x)$ can then be defined as $\phi(x) = \underbrace{\min d}_{x_\Gamma \in \Gamma} sign(n.(x - x_\Gamma))$ where n the unit normal vector.

In level set method (Stolarska, 2001), the interface of interest is represented as the zero level set of a function $\phi(x(t),t)$. This function is one dimension higher than the dimension of the interface. The evolution equation for the interface can then be expressed as an equation for the evolution of ϕ . The level set curve of a function $\phi(x,t): R^2 \times R \to R$, where $\gamma(t) = \{x \in R^2 : \phi(x,t) = 0\}$. The motion of $\gamma(t)$ is translated into an evolution equation for ϕ by taking the time derivative of $\phi(x(t),t) = 0$. The sign of the minimum distance depends on which side of the interface a point x is located. A crack is represented as the zero level of a function $\psi(x,t)$. The end points of the crack will be represented by the intersection of the zero level sets of two functions, $\psi(x,t)$ and $\phi_i(x,t)$, where the subscript i corresponds to the i^{th} end point.

2.2. Geometric Description of Crack

We model one dimensional crack growth in a level set framework by representing the crack as the zero level set of a function $\psi(x,t)$. An end point of the crack is represented as the intersection of the zero level set of ψ with an orthogonal zero level set of the function $\phi_i(x,t)$, where i is the number of tips on a given crack. The level set function representing the initial crack is constructed by computing the signed-distance function for the crack. The initial level set functions, ψ and ϕ_i, the representation of the crack are shown in Figure 1 (Stolarska, 2001).

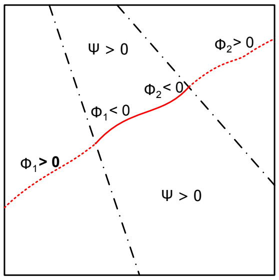

Figure 1. Geometric Description of a Crack.

An important consideration is the actual crack is embedded inside a domain, the zero level set of ψ cuts through the entire domain. For the case of more than one crack tip, it is convenient to define a single function

$\phi(x,t)$ for the crack tip level set representation by $\phi(x,t)=\max\limits_{i}(\phi_i)$ and hence a crack is defined as the set $\{x:\psi(x,t)=0 \;\; and \;\; \phi(x,t)\le 0\}$.

3. EXTENDED FINITE ELEMENT METHOD(XFEM)

First, a finite element method (FEM), i.e., a mesh and associated nodal shape functions, is built while ignoring the crack. Then, based upon the precise geometry of the crack, simple, auxiliary nodal shape functions are added to some of the existing nodes. The solutions of the equations can thus naturally provide a discontinuity in displacement at all points along the crack. The main appeal of XFEM is that it can model, without any remeshing, the deformations due to cracks of arbitrary shapes and also the way they propagate through matter. Since XFEM can be viewed as an extension of FEM, one should be able to add XFEM capabilities to existing FEM frameworks. The finite element shape functions form a partition of unity $\sum_{I\in N} N_I(x)=1$. It follows from the above that for an arbitrary function $\varphi(x)$, the following satisfies $\sum_{I\in N} N_I(x)\varphi(x)=\varphi(x)$. Therefore any function φ can be reproduced by a set of functions $N_I\varphi$. This is the key property of enriched finite element methods based on a partition of unity (Jack, 2003). Additional enrichment approximation is added to the classical finite element model to account for the effects of a crack or discontinuity.

4. AN ANALYSIS OF XFEM FOR MODELING CRACKS

We construct XFEM from case by case to determine the approximation of discontinuous displacement field and now consider the two main cases which are crack is aligned with a mesh and crack is not aligned with mesh. We first consider a simple case of an edge crack modeled by four elements as shown in Figure 2. In this case, it can be divided into three cases. In these cases, the position of the tip will be neglectgible. By XFEM, a crack arbitrarily aligned within the mesh can be represented by means of enrichment functions (Christopher, 2000). In Figure 2, the local co-ordinate system is aligned with the crack. We wish to illustrate how an equivalent discrete space can be constructed with the mesh shown in Figure 3 and the addition of a discontinuous field.

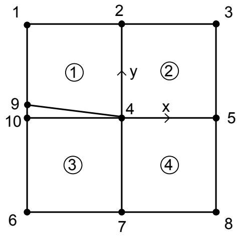

Figure 2. FEM Mesh.

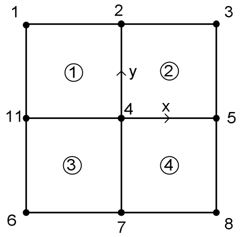

Figure 3. Extended Finite Element Mesh.

The finite element approximation associated with the mesh in Figure 2 is

$$u^h=\sum_{i=1}^{10} u_i N_i \tag{1}$$

where u_i is the displacement at node i and N_i is the shape function associated with node i. Each shape function N_i has a compact support ω_i given by the union of the elements connected to node i. Let us introduce the nodal variables α_{11} and β_{11} ; $\alpha_{11} = \frac{u_9 + u_{10}}{2}$ and $\beta_{11} = \frac{u_9 - u_{10}}{2}$. We can express u_9 and u_{10} in terms of α_{11} and β_{11}; $u_9 = \alpha_{11} + \beta_{11}$, $u_{10} = \alpha_{11} - \beta_{11}$. Replacing u_9 and u_{10} in terms of α_{11} and β_{11}, we have

$$u^h = \sum_{i=1}^{8} u_i N_i + \alpha_{11}(N_9 + N_{10}) + \beta_{11}(N_9 + N_{10}) H_I(x). \tag{2}$$

The function $H_{\mathrm{I}}(x,y) = \begin{cases} 1 & for\ y > 0 \\ -1 & for\ y < 0 \end{cases}$, such that $H_{\mathrm{I}}(x) = 1$ on element 1 and -1 on element 3, is a discontinuous or enrichment function defined in the local crack co-ordinate system as shown in Figure 4.

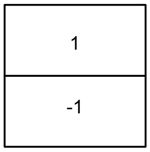

Figure 4. Enrichment Function $H_{\mathrm{I}}(x)$.

If we now consider the mesh in Figure 3, $N_9 + N_{10}$ can be replaced by N_{11} and α_{11} by u_{11}. The approximation is

$$u^h = \sum_{i=1}^{8} u_i N_i + u_{11} N_{11} + \beta_{11} N_{11} H_{\mathrm{I}}(x). \tag{3}$$

The first two terms on the right-hand side represent the classical finite element approximation, whereas the last one represents the addition of a discontinuous enrichment. When a crack is modeled by a mesh in Figure 2, we may interpret the finite element space as the sum of one which does not model the crack (such as Figure 3) and a discontinuous enrichment. In terms of enrichment with the jump function, we adopt the convention crack into two disjoint pieces. This rule is seen to be consistent in which only node 11 was enriched. So in this case, there are 9 nodes and the extended finite element method of the approximation of displacement field is

$$u^h = \sum_{i=1}^{9} u_i N_i + \sum_{j=1}^{1} \beta_j N_j H_{\mathrm{I}}(x) \tag{4}$$

where $i \in I$ is the set of all nodes in the mesh, u_i is the displacement (or the classical degree of freedom) at node i, N_i is the shape function associated with node i. Each shape function N_i has compact support ω_i given by the union of the elements connected to node i. $j \in J, J \subset I$ is the subset of nodes that are enriched for the crack discontinuity and β_j are the corresponding additional degrees of freedom in J are such that their support, we mean the support of the nodal shape function intersects the crack but do not contain any of its crack tips.

In this case, we introduce the discontinuous functions for modeling branched cracks as shown in Figure 5 and Figure 6 by enriching with the discontinuous function $H(x)$. In this case, the numbers of nodes are 14.

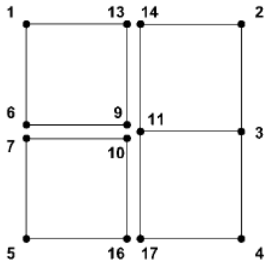

Figure 5. FEM Mesh.

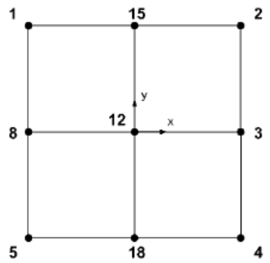

Figure 6. XFEM Mesh.

The nodal variables $\alpha_8, \beta_8, \alpha_{15}, \beta_{15}, \alpha_{18}, \beta_{18}$ are considered the same as in the previous case, $\alpha_{12} = \dfrac{u_{11} + {(u_9 + u_{10})}/{2}}{2}$, $\beta_{12} = \dfrac{u_{11} - {(u_9 + u_{10})}/{2}}{2}$, $\gamma_{12} = \dfrac{u_9 - u_{10}}{2}$ However, $u_6, u_7, u_{14}, u_{16}, u_{11}, u_{13}, u_{17}$ can be defined exactly the same as in the previous case. The other displacement u_9 and u_{10} are needed to define here as follow. Again, $u_6, u_7, u_{14}, u_{16}, u_{11}, u_{13}, u_{17}$ can be expressed in terms of nodal variable mentioned above. u_9 and u_{10} can be expressed as $u_9 = \alpha_{12} - \beta_{12} + \gamma_{12}$, $u_{10} = \alpha_{12} - \beta_{12} - \gamma_{12}$. Now by expressing u_i in terms of α_j , β_j, γ_j, in equation (1), we obtain the extended finite element method of the approximation of displacement field ,

$$\begin{aligned} u^h = \sum_{i=1}^{5} u_i N_i + \alpha_8(N_6 + N_7) + \beta_8 H_I(x)(N_6 + N_7) + \alpha_{12}(N_9 + N_{10} + N_{11}) + \beta_{12} H_{II}(x)(N_9 + N_{10} + N_{11}) \\ + \gamma_{12} J(x)(N_9 + N_{10}) + \alpha_{15}(N_{13} + N_{14}) + \beta_{15} H_{II}(x)(N_{13} + N_{14}) + \alpha_{18}(N_{16} + N_{17}) + \beta_{18} H_{II}(x)(N_{16} + N_{17}). \end{aligned} \tag{5}$$

$H_{\mathrm{I}}(x)$ has been defined in previous case. The other $H_{II}(x)$ and $J(x)$ can be defined as follows shown in Figure 7.

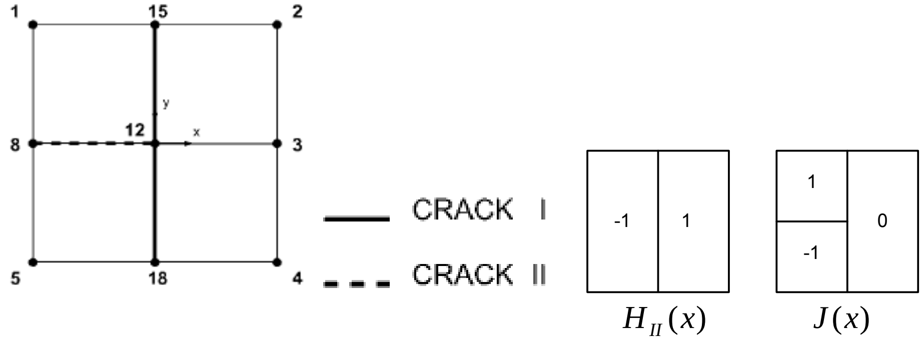

Figure 7. Definition of the Discontinuous Enrichment Functions $H_{II}(x)$ and $J(x)$.

Consider the branched crack as consisting of two cracks denoted by crack I and II in Figure 7. Deciding that crack I is parameterized so that we transverse it from nodes 8 to 12, the functions $H_I(x)$ and $H_{II}(x)$ are then the function $H(x)$ associated with cracks I and II, respectively. The function $J(x)$ is referred to as a discontinuous 'junction' function. It may be expressed in terms of the functions $H_I(x)$ and $H_{II}(x)$, $J(x) = \begin{cases} H_I(x) & for\ H_{II}(x) < 0 \\ 0 & for\ H_{II}(x) > 0 \end{cases}$. So the extended finite element approximation is

$$u^h = \sum_{i=1}^{9} u_i \phi_i + \sum_{j=1}^{5} \beta_j N_j H(x). \tag{6}$$

Having introduced the junction function *J(x)*, we now describe how we can extend it to model multiple branched cracks. In the case of crossing cracks, the first idea that comes to mind is to consider the two cracks as independent cracks as shown in Figure 8 and 9. There are 16 nodes. Therefore

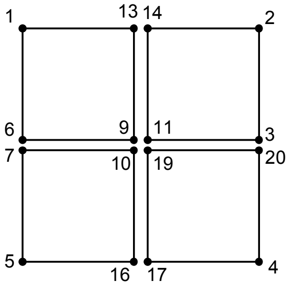

Figure 8. FEM mesh.

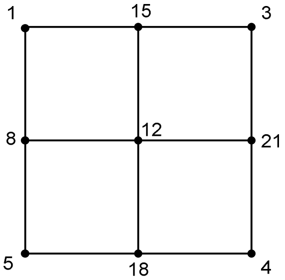

Figure 9. XFEM mesh.

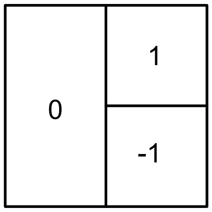

Figure 10. $J'(x)$.

$$\begin{aligned} u^h = {} & \sum_{i=1}^{4} u_i N_i + \alpha_8 (N_6 + N_7) + \beta_8 H_I(x)(N_6 + N_7) + \alpha_{15}(N_{13} + N_{14}) + \beta_{15} H_{II}(x)(N_{13} + N_{14}) + \alpha_{18}(N_{16} + N_{17}) \\ & + \beta_{18} H_{II}(x)(N_{16} + N_{17}) + \alpha_{21}(N_3 + N_{20}) + \beta_{21} H_I(x)(N_3 + N_{20}) + \alpha_{12}(N_9 + N_{10} + N_{11} + N_{19}) \\ & + \beta_{12} H_{II}(x)(N_9 + N_{10} + N_{11} + N_{19}) + \gamma_{12} J(x)(N_9 + N_{10}) + \tau_{12} J'(x)(N_{11} + N_{19}). \end{aligned} \tag{7}$$

The function $J'(x)$ is referred to as a discontinuous 'junction' function. It may be expressed in terms of the functions $H_I(x)$ and $H_{II}(x)$, $J'(x) = \begin{cases} H_{II}(x) & for\ H_I(x) < 0 \\ 0 & for\ H_I(x) > 0 \end{cases}$. The approximation can now be written as

$$u^h = \sum_{i=1}^{9} u_i N_i + \sum_{j=1}^{6} \beta_j N_j H(x). \tag{8}$$

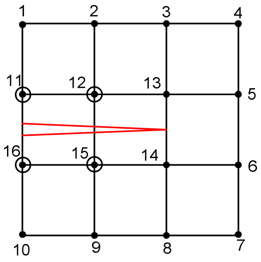

Figure 11. Crack not Aligned with a Mesh.

In Figure 11, the circled nodes are enriched with the discontinuous function. In this case, the tip will be neglectgible. The circled nodes are enriched with the jump function. According to the discretization shown in Figure 11, the approximation of the displacement field is

$$u^h = \sum_{i=1}^{16} u_i N_i + \sum_{j=1}^{4} \beta_j N_j H(x). \tag{9}$$

Here $H(x)$ is the modified Heaviside function which takes on the value +1 above the crack and -1 below the crack. We now generalize to the case of an arbitrary crack, as shown in Figure 12.

In a more general case as shown in Figure 12, the circled nodes are enriched with the discontinuous function and the squared nodes with the tip enrichment functions. Enrichment with only the discontinuous function shortens the crack to point *p*, the crack tip will not coincide with an element edge, and in this instance the discontinuity cannot be adequately described using only a function such as $H(x)$. The jump enrichment of the circled nodes in this case only provides for the modeling of the discontinuity up until point *p* (Nicolas, 1999).

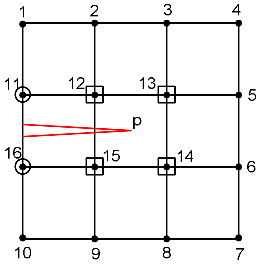

Figure 12. Crack not Aligned with a Mesh

To seamlessly model the entire with along the crack, the square nodes are enriched with the asymptotic crack tip functions with the technique. As the discretization shown in Figure 12, the approximation is

$$u^h = \sum_{i=1}^{16} u_i N_i + \sum_{j=1}^{2} \beta_j N_j H(x) + \sum_{k=1}^{4} N_k \left(\sum_{l=1}^{4} c_k^l F_l(x) \right) \tag{10}$$

in which $j \in J$, J is the set of circled nodes and $k \in K$, K is the set of squared nodes. The functions $F_l(x)$ are defined as $F_l(r,\theta) = \left\{ \sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\theta\sin\frac{\theta}{2}, \sqrt{r}\sin\theta\cos\frac{\theta}{2} \right\}$, where (r,θ) are the local polar co-ordinates at the crack tip. Note that the first function $\sqrt{r}\sin(\frac{\theta}{2})$ is discontinuous across the crack faces whereas the last three functions are continuous.

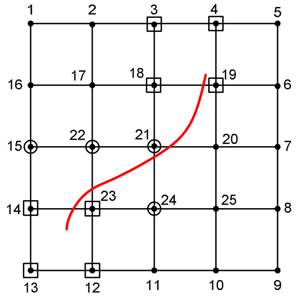

Figure 13. Crack with Two Tips

This case is crack not aligned with mesh and two tips. The approximation is

$$u^h = \sum_{i=1}^{25} u_i N_i + \sum_{j=1}^{6} \beta_j N_j H(x) + \sum_{\substack{k=1 \\ k \in K_1}}^{4} N_k \left(\sum_{l=1}^{4} c_k^{l1} F^1{}_l(x) \right) + \sum_{\substack{k=1 \\ k \in K_2}}^{4} N_k \left(\sum_{l=1}^{4} c_k^{l2} F_l{}^2(x) \right) \tag{11}$$

where K_1 and K_2 are the sets of nodes to be enriched for the first and second crack tip, respectively. The precise definition of these two sets as well as the set J will be given further. The functions $F_l^1(x)$ and $F_l^2(x)$ are identical to the ones given in (11), with (r_1, θ_1) and (r_2, θ_2) being defined in the local crack tip system at tips 1 and 2, as shown in Figure 13.

5. CONCLUSION

A method has been developed for modeling crack growth by enrichment that includes the asymptotic near tip field. The method treats the crack as a completely separate geometric entity and the only interaction with the mesh occurs in the selection of the enriched nodes. In contrast to the element enrichment, the accuracy of this enrichment based on a partition of unity is almost independent of element size for a large range. In addition, the use of crack tip elements requires transition elements. The need of a variable number of degrees of freedom is a factor that contributes to greater computational cost of the XFEM.

We have analysed and reported several cases of positions of cracks and meshes. There are three kinds of sets of nodes, *I, J, K.* According to the selection of the enriched nodes, the set *J* occurs and needs to define enrichment function *H* for those enriched nodes. Similarly, due to the tip, the set *K* occurs and needs to define the function *F*.

6. FURTHER EXTENSIONS

The crack growth and crack direction can be approximated from the determination and classification of the approximation of the discontinuous displacement field. In a future study, we are going to investigate how to apply XFEM for a crack growth and crack direction, and the numerical examples for the crack problems will be illustrated to highlight the real application of XFEM.

7. ACKNOWLEDGEMENT

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